

CHAOTIC BOLTZMANN MACHINES WITH TWO ELEMENTS

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ABSTRACT. In this brief note, we show that chaotic Boltzmann machines truly yield samples from the probabilistic distribution of the corresponding Boltzmann machines if they are composed of only two elements. This note is an English translation (with slight modifications) of the article originally written in Japanese [H. Suzuki, *Seisan Kenkyu* 66 (2014), 315–316].

1. INTRODUCTION

Recently, chaotic Boltzmann machines [1] were proposed as a deterministic implementation of Boltzmann machines. The apparently stochastic behavior of chaotic Boltzmann machines is achieved, without any use of random numbers, by chaotic dynamics that emerges from pseudo-billiard dynamics. It was shown numerically that the chaotic billiard dynamics of chaotic Boltzmann machines can be used for generating sample sequences from the probabilistic distribution of Boltzmann machines, and it was successfully applied to other spin models such as the Ising model and the Potts model [2].

Despite these numerical evidences, there have been no theoretical proof that chaotic Boltzmann machines yield samples from the probabilistic distribution of the corresponding Boltzmann machines.

In this brief note, as a first step of theoretical approach, we investigate the simplest system. Namely, we show that chaotic Boltzmann machines truly yield samples for the corresponding Boltzmann machines if they are composed of only two elements. Although our approach cannot be applied to larger chaotic Boltzmann machines with more than two elements, we expect that it gives some insights into the dynamics of larger chaotic Boltzmann machines.

Since the proof is not entirely trivial, it is considered worth making available on arXiv. This note is an English translation (with slight modifications) of the article [3] originally written in Japanese.

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2. CHAOTIC BOLTZMANN MACHINES WITH TWO ELEMENTS

Let S_1 and S_2 be random variables that take values on $\{0, 1\}$. We assume that the probabilistic model $P[S_1, S_2]$ is given in the form of conditional probabilities $P[S_1 | S_2]$ and $P[S_2 | S_1]$, which is common in Boltzmann machines and many other spin models in statistical physics.

Let us consider a chaotic Boltzmann machine for the probabilistic model. In addition to the states s_1 and $s_2 \in \{0, 1\}$ of the elements, we introduce internal states x_1 and $x_2 \in [0, 1]$ of the elements. Therefore, the state space of the chaotic Boltzmann machine is $\{0, 1\}^2 \times [0, 1]^2$. The internal states x_1 and x_2 evolve according to the following differential equations:

$$(1) \quad \frac{dx_1}{dt} = (1 - 2s_1)C_1(s_2)P[s_1 | s_2]^{-1},$$

$$(2) \quad \frac{dx_2}{dt} = (1 - 2s_2)C_2(s_1)P[s_2 | s_1]^{-1},$$

where $C_i(\cdot) > 0$ represents arbitrary positive constants defined for each state of the other element. When the internal state x_i reaches 0 or 1, the state s_i of the element changes as follows:

$$(3) \quad s_i \longleftarrow 1 \quad \text{when} \quad x_i = 1,$$

$$(4) \quad s_i \longleftarrow 0 \quad \text{when} \quad x_i = 0.$$

The internal state (x_1, x_2) moves straight inside the unit square $[0, 1]^2$, and changes its direction when it hits a side of the square. Therefore, this system can be understood as a pseudo-billiard in the billiard table $[0, 1]^2$.

What we expect here is that the state variables s_1 and s_2 observed from this system follow the distribution $P[S_1, S_2]$. Namely, our goal in this note is to show the following proposition.

Proposition 1. *For any initial values $(s_1(0), s_2(0), x_1(0), x_2(0)) \in \{0, 1\}^2 \times [0, 1]^2$, the chaotic Boltzmann machine defined by Eqs. (1)–(4) satisfies*

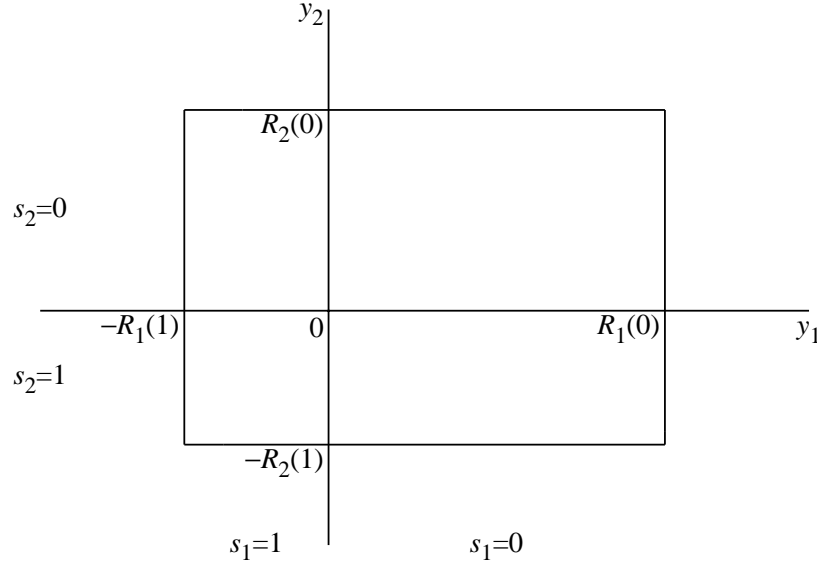
$$(5) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \delta(s_1(t), s'_1) \delta(s_2(t), s'_2) dt = P[s'_1, s'_2]$$

for $(s'_1, s'_2) \in \{0, 1\}^2$, provided that $\frac{R_2(0) + R_2(1)}{R_1(0) + R_1(1)}$ is irrational. Here, $\delta(\cdot, \cdot)$ is Kronecker's delta function, and $R_i(s_i) = C_{3-i}(s_i)P[s_i]$.

3. PROOF: QUASI-PERIODIC DYNAMICS

To show this proposition, we rewrite Eqs. (1) and (2) of the chaotic Boltzmann machine. Specifically, instead of the state (s_i, x_i) of each element, we introduce a new state variable $y_i = H_i(s_i, x_i)$ using the map

$$(6) \quad H_i(s_i, x_i) = (1 - 2s_i)R_i(s_i)x_i.$$

FIGURE 1. The state space for (y_1, y_2) .

Then the state space $\{0, 1\}^2 \times [0, 1]^2$ of the chaotic Boltzmann machine is mapped to the rectangular $[-R_1(1), R_1(0)] \times [-R_2(1), R_2(0)]$ (see Fig. 1). On this state space, the dynamics of the chaotic Boltzmann machine (Eqs. (1)–(4)) is rewritten as follows:

$$(7) \quad \frac{dy_1}{dt} = R_1(s_1)C_1(s_2)P[s_1|s_2]^{-1} = \frac{R_1(s_1)R_2(s_2)}{P[s_1, s_2]},$$

$$(8) \quad \frac{dy_2}{dt} = R_2(s_2)C_2(s_1)P[s_2|s_1]^{-1} = \frac{R_1(s_1)R_2(s_2)}{P[s_1, s_2]},$$

and

$$(9) \quad y_i \longleftarrow -R_1(1) \quad \text{when} \quad y_i = R_1(0),$$

$$(10) \quad y_i \longleftarrow -R_2(1) \quad \text{when} \quad y_i = R_2(0),$$

where the state s_i can be determined from y_i as $s_i(y_i) = 0$ when $y_i \geq 0$, and $s_i(y_i) = 1$ when $y_i < 0$.

The differential equations (7) and (8) always satisfy $\frac{dy_1}{dt} = \frac{dy_2}{dt}$. Hence, the state (y_1, y_2) moves in the direction of $(1, 1)$ in the rectangular state space. Here we introduce (z_1, z_2) which moves along the orbit of (y_1, y_2) at a constant velocity $\frac{dz_1}{dt} = \frac{dz_2}{dt} = 1$. Then the orbit of (z_1, z_2) is equidistributed in the state space in the following sense. Let us consider the Poincaré section on $z_1 = -R_1(1)$. While the value of z_1 changes by $(R_1(0) + R_1(1))$, the

value of z_2 changes by the same amount. Therefore, the Poincaré map on $z_1 = -R_1(1)$ is a rigid rotation with rotation number $(R_2(0) + R_2(1))/(R_1(0) + R_1(1)) \bmod 1$. If the rotation number is irrational, the Poincaré map is ergodic with respect to the uniform distribution. Then the orbit is also equidistributed in the state space, and we have

$$(11) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \delta(s_1(z_1(t)), s'_1) \delta(s_2(z_2(t)), s'_2) dt \propto R_1(s'_1) R_2(s'_2).$$

Since (y_1, y_2) moves along the same orbit as (z_1, z_2) with the velocity described in Eqs. (7) and (8), we have

$$(12) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \delta(s_1(y_1(t)), s'_1) \delta(s_2(y_2(t)), s'_2) dt = P[s'_1, s'_2],$$

which completes the proof.

4. CONCLUDING REMARKS

In this brief note, we have shown that chaotic Boltzmann machines with two elements have quasi-periodic dynamics and yield samples from the probabilistic distribution of the corresponding Boltzmann machines, provided that the rotation number is irrational.

The set of parameter values $C_i(\cdot)$ that make rotation numbers rational has Lebesgue measure zero. This is analogous to probabilistic Monte Carlo sampling, which does not work with probability (measure) zero. However, when we design chaotic Boltzmann machines artificially, it may be possible that the rotation number easily becomes rational. In such a case, we have to adjust $C_i(\cdot)$ by multiplying some constants to make the rotation number irrational.

The performance as an sampling algorithm depends on the characteristics of the rotation number. At least, a chaotic Boltzmann machine with a good rotation number that yields a low-discrepancy sequence is expected to exhibit better performance than the corresponding Boltzmann machine.

Our approach in this note cannot be applied to larger chaotic Boltzmann machines with more than two elements, which exhibit chaotic behavior. However, some aspects are expected to be shared with larger systems; for example, our preliminary numerical study shows that irrationality is important, for chaotic Boltzmann machines composed of not many but more than two elements, to generate faithful sample sequences. Therefore, we expect that our approach gives some insights into the dynamics of larger chaotic Boltzmann machines.

REFERENCES

- [1] Hideyuki Suzuki, Jun-ichi Imura, Yoshihiko Horio, and Kazuyuki Aihara: Chaotic Boltzmann machines, *Scientific Reports* 3 (2013), 1610.
- [2] Hideyuki Suzuki: Monte Carlo simulation of classical spin models with chaotic billiards, *Physical Review E* 88 (2013), 052144.
- [3] Hideyuki Suzuki: Chaotic Boltzmann machines with two elements, *Seisan Kenkyu* 66 (2014), 315–316 (in Japanese).

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